3:1 Nesting Rules in Redistricting

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Abstract

In legislative redistricting, most states draw their House and Senate maps separately. Ohio and Wisconsin require that their Senate districts be made with a 3:1 nesting rule, i.e., out of triplets of adjacent House districts. We study the impact of this requirement on redistricting, specifically on the number of seats won by a particular political party. We compare two ensembles generated using Markov Chain Monte Carlo methods; one which uses the ReCom chain to generate Senate maps without a nesting requirement, and the other which uses a chain that generates Senate maps with a 3:1 nesting requirement. We find that requiring a 3:1 nesting rule has minimal impact on the distribution of seats won. Moreover, we probe how 3:1 nesting can mitigate partisan gerrymandering, and find that nesting reduces the ability of a party to bias the Senate map.

Keywords: Markov chains, ensemble analysis, gerrymandering

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1 Introduction

Each state in the U.S., with the exception of Nebraska, has a bicameral legislature, meaning there is an upper and lower house [2]. Most states draw district maps for their two houses separately. Eight states require that the state Senate map be made of pairs of adjacent districts from the lower house; we call this a *2:1 nesting rule*. Two states, Ohio and Wisconsin, require a *3:1 nesting rule*, in which the state Senate map is made of triplets of adjacent districts from the lower house. These rules are often in place to help election administration; when you know that everyone in a house district is a member of the same senate district, you have to print fewer ballot types.

In theory, these nesting rules severely restrict the number of possible maps that can be drawn. Consider the toy example on the 6×6 grid in Figure 1 from [10]. There are 264,500

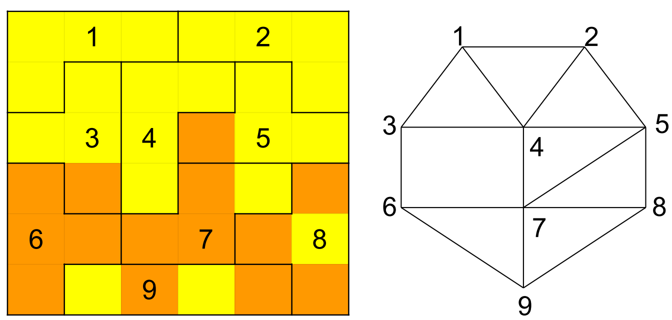

Figure 1: The 6×6 grid, where each square represents one voter, either of the yellow or orange party. We have chosen a fixed partition into nine House districts and constructed the corresponding dual graph.

ways to construct a Senate plan with 3 districts with no restrictions [1]. However, if we fix a lower house map with nine districts, and only consider Senate maps made out of triplets of adjacent districts, we find there are only 14 Senate plans. Common wisdom says that this restriction on the space of redistricting plans should reduce the variability of the plans. But even in this toy example, we see that under the matching $\{(1,2,4),(3,6,9),(5,7,8)\}$, the yellow party wins one seat, while in the matching $\{(1,3,6),(2,5,8),(4,7,9)\}$, yellow wins all three seats. This suggests that even with a restricted state space, a significant variation in election outcome is possible.

In this paper, we analyze the impact of these 3:1 nesting rules on redistricting. Our key findings are twofold. First, that a 3:1 nesting rule has little impact on the distribution of Democrat seats won when compared to unnested maps. Second, that nesting allows a wide range of Senate maps regardless of bias at the House level, as well as reduces the overall ability of a party to optimize for seats won. This suggests that as a policy intervention, nesting may be used to mitigate partisan gerrymandering.

The rest of the paper proceeds as follows. Section 2 introduces the basic mathematics of redistricting and sampling plans with Markov chains. In Section 3, we provide an overview of prior work and introduce the Swap Markov chain which we use to sample 3:1 nested maps. Section 4 discusses the provenance of our election and geospatial data, while Section 5 examines convergence diagnostics of our Markov chains. In Section 6, we compare the properties of nested and unnested maps and in Section 7, we study the ability of nesting to mitigate gerrymandering. Finally, Section 8 proposes some future directions for study and policy implications of our work.

2 The Mathematics of Redistricting

2.1 Dual Graphs

Redistricting is a discrete problem. States are divided into small units, like precincts or Census blocks, and these discrete units are then assigned to districts. Mathematically, we formulate this problem using a *dual graph*. The vertices of the dual graph G are the discrete units, and an edge denotes geographic adjacency, where we require that two units share a border of positive length. A *districting plan* $D = (D_1, \ldots, D_n)$ is a partition of the vertices of the dual graph into n components, i.e., districts. By both federal and state law, districting plans must meet a host of other requirements. Some of the most typical ones include

1. contiguity; each component of the partition must form a connected induced subgraph of G. That is, districts must be connected.

2. population balance; since the U.S. Supreme Court ruling in Reynolds v. Sims in 1964 [17], districts must have roughly equal population. We usually take this to be within 5% of the ideal population, but map drawers usually can balance this to within just a few individuals [13].

3. compactness; this is hard to operationalize, but people do not like seeing "snakey" districts.

We take our dual graph to be the *House dual graph*, in which the vertices represent House districts. A *3:1 nested Senate map* is thus a districting plan on the House dual graph with the further constraints that each component of the partition be of size 3 and each component be connected. We do not take population balance into account; we assume that the underlying House dual graph is already population balanced. When making nested maps, we assume that the chosen House map is compact, and do not impose further compactness restrictions on the Senate maps.

2.2 Ensemble Analysis

Gerrymandering is the process of drawing districts to advantage one class of people over another. The question of how to tell if a map is gerrymandered goes well beyond the scope of this paper and crosses numerous academic disciplines and practical issues. One prominent method in the literature is the use of *ensemble methods* (see Chapter 16 and 17 of [9]). Using some generative process, a large number, or *ensemble*, of districting plans are constructed for a particular state. Then, a proposed or enacted map can be compared to the ensemble, and if it is an outlier in some statistic, that might indicate that it is gerrymandered.

One generative method used in the redistricting literature for sampling graph partitions is to use Markov Chain Monte Carlo (MCMC) methods (see, for example, [8]) . Informally, a Markov chain is a process that takes an initial object, updates that object via some probabilistic rule, and returns the new object. This is one step of the chain; MCMC methods involve taking many steps in the chain, and using the generated objects as a sample. The ReCom Markov chain was first introduced in [8] to generate districting plans that meet the generally accepted standards for "good" districts. Given a dual graph and an initial districting plan, the ReCom chain merges two adjacent districts, generates a spanning tree on the resulting induced subgraph, and randomly cuts the spanning tree, resulting in two new districts.

We use the ReCom algorithm on the precinct/ward dual graphs for Ohio/Wisconsin to generate ensembles of Senate plans that do not follow any 3:1 nesting rule (unnested Senate plans). To generate maps that follow a 3:1 nesting rule (nested Senate plans), we use the enacted House plan as our dual graph and run a Markov chain called "Swap", which is discussed in Section 3.1.

3 Prior Work

Constructing maps using nesting rules has been proposed as a game-theoretic way of generating fair maps [14]. In [4], the authors explore how 2:1 nesting rules impact redistricting. They found that in Alaska, the 2:1 nesting rule had minimal impact on how many seats Democrats could win. In other words, they could win the same range of seats with or without the 2:1 nesting rule.

In [4], the authors are able to do two kinds of analysis. When computationally feasible, they construct all possible 2:1 nestings on a House dual graph, and study the properties of the maps. When it is infeasible to construct all 2:1 nestings, they can uniformly sample them thanks to a connection to perfect matchings and the FKT algorithm. Unfortunately, constructing all possible 3:1 nested plans for Ohio or Wisconsin is computationally infeasible. The 3:1 nesting requirement can be formulated as a problem about perfect matchings on 3-uniform hypergraphs, which is one of Karp's original NP-complete problems [12]. Moreover, the method of uniform sampling used in [4] does not extend to 3:1 nestings.

3.1 Swap Chain

In order to sample 3:1 nested Senate maps, we make use of a Markov chain first introduced by Durham in [10], which we call the *Swap* chain. Given a 3:1 nested Senate map, to take a step in the chain, proceed as follows.

1. Choose two House districts uniformly at random with replacement.

2. Swap the Senate district assignment of the two House districts.

3. If this swap does not create a valid (i.e., contiguous) Senate map, go back to step 1.

4. If the swap does create a valid (contiguous) Senate map, accept the new map.

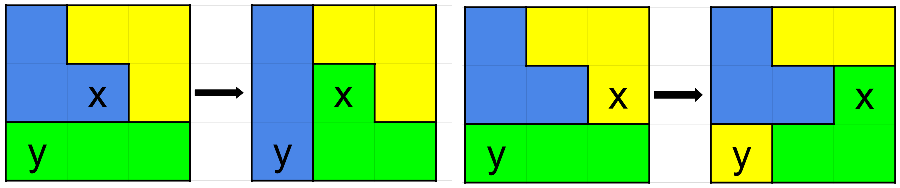

Figure 2: The left side shows a valid swap of two House districts, while the right side shows an invalid swap, since it disconnects the yellow district.

See Figure 2.

The Swap chain is almost equivalent to running ReCom on the House dual graph if you impose a population of 1 on each node and ask ReCom to generate maps with strict population balance. Any swap of two vertices can be achieved by cutting a spanning tree and vice versa. Thus, the two chains have equivalent state space connectivity. A recent arXiv preprint shows Swap is not always irreducible [16]. That is, we cannot always get from one districting plan to every other one via Swap moves. However, this theoretical result is about grid graphs, and we see no evidence that our chain is not effectively moving through the state space for real world dual graphs.

In this setting, ReCom and Swap do sometimes differ in their target measures. Since Swap has symmetric transition probabilities, it targets the uniform distribution. ReCom nearly targets the spanning tree distribution in which the probability of sampling a plan is proportional to the number of spanning trees [8]. In the case of 3:1 nested maps, this would favor districts made of 3-cycles over districts that are made of paths. Since our study is concerned with understanding the distribution of possible 3:1 nested maps, and since we see no real world preference for one kind of Senate nesting over another, we choose to use Swap so that we target the uniform distribution.

4 Our Data

In order to analyze maps produced by our Markov chains, we need to know how people vote. Our best substitute for this is to take past election data and assume that people living in a district will vote the way they did in a previous election. Unfortunately, voting data is reported at the precinct/ward group level in Ohio and Wisconsin, while districts are composed of Census blocks. We thus rely on third party collection sources who put in an enormous amount of effort creating digital SHP files that record the boundaries of precincts/wards along with election data. In this paper, we use both the MGGG Redistricting Lab's (MGGG) and the Redistricting Data Hub's (RDH) collection of SHP files that include precinct boundaries and election data. For more information on which SHP files we used, how we included population data, any preprocessing decisions made before running our Markov chains, and our code for running the chains, see our GitHub repository [6].

In Ohio, out of the nine statewide elections for which we had precinct files, we selected the 2018 Senate (SEN18) and Treasurer (TRES18) races as our election data. The 2018 Senate and Treasurer election data have a very similar vote percentage, but flipped for each party. In SEN18 we have 53.4% for the Democratic candidate and 46.6% for the Republican, while in the TRES18 we have 53.5% for the Republican and 46.7% for the Democrat. There is not a major third party/write-in presence in either race.

In Wisconsin, out of the six statewide elections since 2016 for which we had SHP files, we decided to work with the 2018 Senate (SEN18) and Attorney General (AG18) races. The 2018 Senate race has the lowest Republican vote share out of all of the available races (44.6%), while the 2018 Attorney General race has the highest (48.8%).

5 Exploring Convergence

It is important to ensure that we have taken enough steps of our Markov chain to converge to the underlying distribution. To explore the convergence of our chains, we proceed as in [7] and [11]; we choose relevant summary statistics for our ensembles, project to said statistics, and explore convergence there. Of course, none of this precludes pseudo-convergence. We choose the number of Democrat seats won as our statistic. We use several heuristics to explore convergence:

1. We compute the n-lag autocorrelation of the desired statistic, i.e., the Pearson correlation between the series and itself shifted by n. An autocorrelation of lag n close to 0 indicates no linear correlation between the statistic at one time and n steps in the future. Autocorrelations that quickly decay to 0 indicate a fast-mixing Markov chain [15].

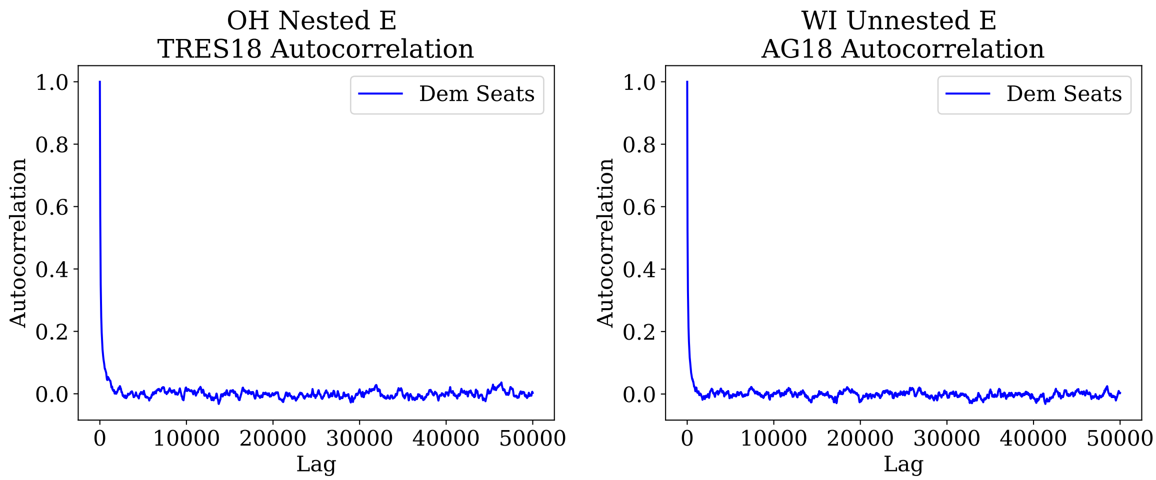

Figure 3: Autocorrelation plots for a 3:1 nested Swap ensemble in Ohio and an unnested ReCom ensemble in Wisconsin, started from the enacted Senate map (E) in each state.

2. We examine the first 10%, 50%, and 100% of the run of the chain, and see if our desired statistic has stabilized.

3. We examine runs of the chain started from different seeds, and see if the desired statistic is reasonably similar across seeds.

We also note that we do not show each figure with every type of election data or for each type of chain. However, all of our results are consistent across such choices unless otherwise noted, and the full set of figures can be found in our GitHub repository [6]. As a result of our discussion below, we run our Swap and ReCom chains for 1 million steps.

For both chains, by 2,000 steps, the autocorrelation for seats won by Democrats begins to hover around 0. See Figure 3. In both Ohio and Wisconsin, we see that 50%, and 100% of a 1 million step ensemble have nearly identical distributions of ranked % Democratic vote share for each chain type. See Figure 4.

In the 3:1 nested setting, a different seed is an alternative Senate map on the same House map. We make use of a method included in the gerrychain Python package that generates random initial seeds; we call these randomly generated seeds S1 and S2. In both states, under any election data, we see very similar histograms of seats won regardless of the starting seed. This is a strong indication of convergence. See Figure 5.

In the unnested setting, a different seed is just an alternative Senate map on the precincts/wards.

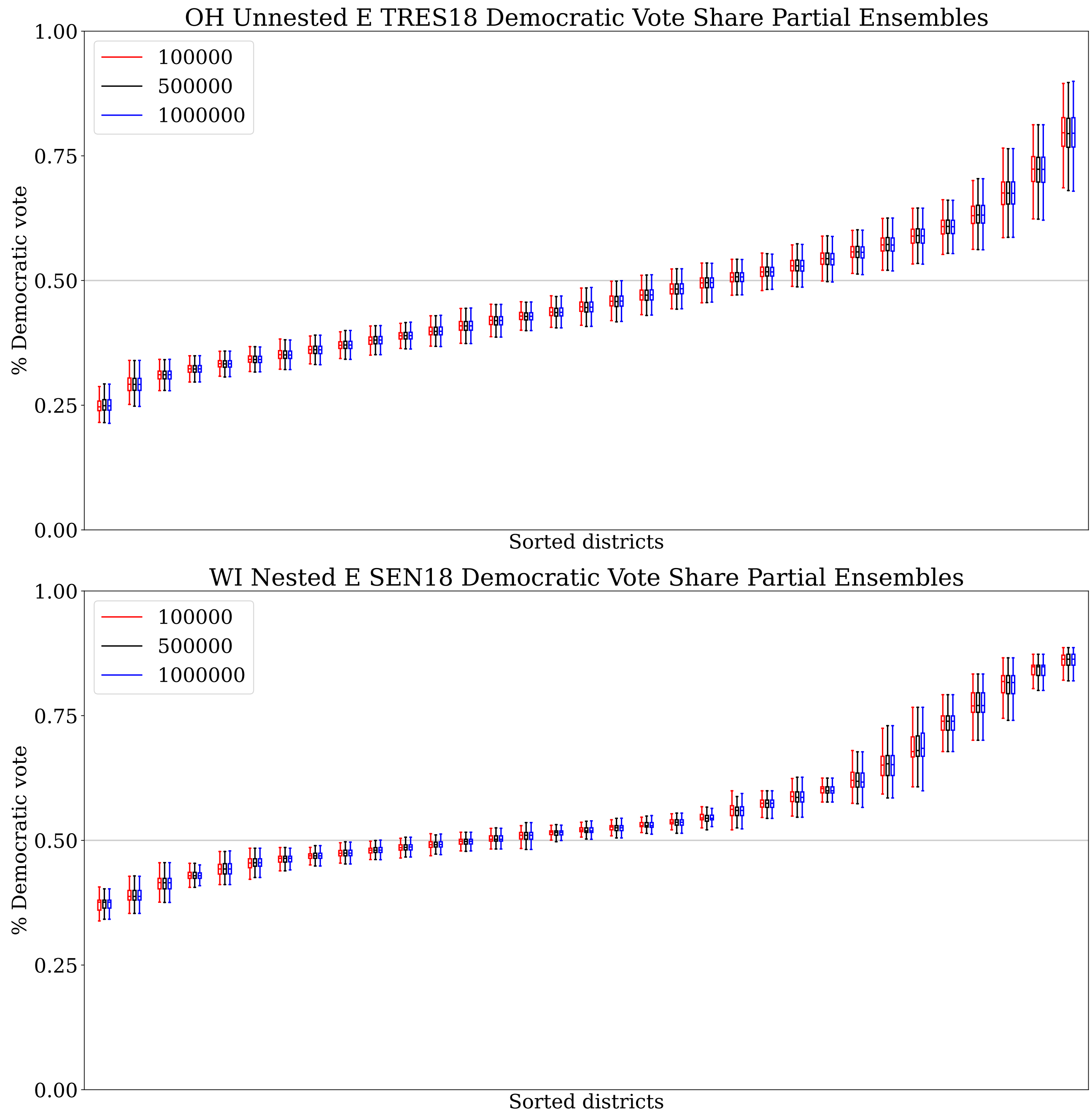

Figure 4: Ranked % Democratic Vote Share in Ohio for 10%, 50%, and 100% of a 1 million step unnested ReCom ensemble and in Wisconsin for a 3:1 nested Swap ensemble, started from the enacted Senate map (E) in each state.

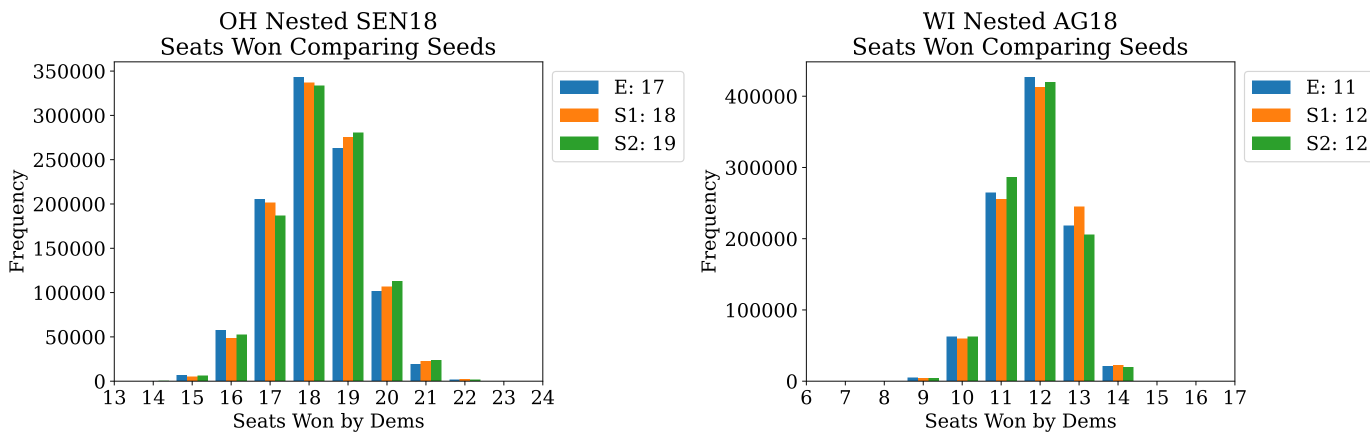

Figure 5: The number of seats won by Democrats across different seed plans for Ohio and Wisconsin 3:1 nested Swap ensembles. The number next to the seed name indicates the number of seats won by Democrats under the seed plan.

The Ohio Redistricting Commission posts submitted Senate plans in a digital format, thus allowing us to have different seeds for the unnested chain. We take the Sykes/Russo Democratic plan (D), the Johnson McDonald Independent plan (I), and the Ohio Citizens' Redistricting Commission plan (C) as three alternate seeds for ReCom. For Wisconsin, alternate Senate plans are not submitted in a format that is easy to digitize, so we again use gerrychain to construct two new seeds for Wisconsin (S1, S2). In Ohio and Wisconsin, we see that our choice of seed for ReCom also does not impact the distribution of seats won by Democrats. This is a strong indication of convergence. See Figure 6.

6 Results

6.1 Comparing Ohio Nested and Unnested

We now compare the behavior of the two ensembles to each other. In some sense, we are taking the unnested ensemble to be the "control" and the nested ensemble tells us how much the 3:1 nesting rule impacts the outcome.

Under SEN18, the nested and unnested ensembles behave almost identically. As seen in Figure 7, the only visible difference is that the nesting requirement seems to shift the distribution marginally to the left, in favor of Republicans. But this shift is negligible in

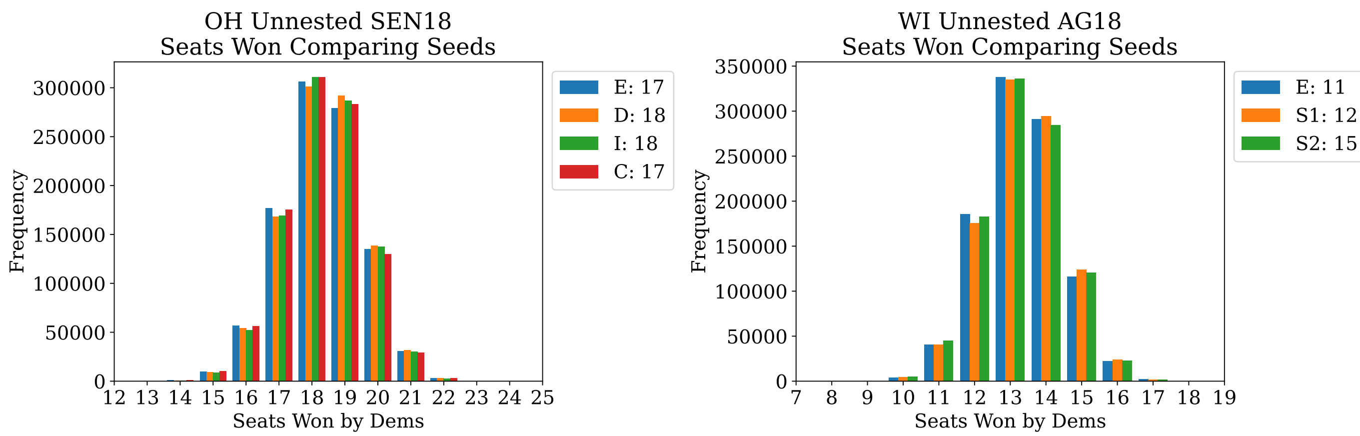

Figure 6: The number of seats won by Democrats across different seed plans in unnested ReCom ensembles. The number next to the seed name indicates the number of seats won by Democrats under the seed plan.

terms of the number of seats; the nested ensemble has a range of 14 to 23 seats, while the unnested ensemble has a range of 13 to 24 seats. While the nesting requirement does narrow the distribution of seats won by one seat in each direction, the unnested ensemble hardly ever samples the 13 and 24 seat maps. In other words, the bulk of the distributions is the same.

Under TRES18, the two ensembles also behave almost identically, except now the nesting requirement shifts the distribution of seats won slightly to the right. See Figure 7. The nested ensemble has a range of 8 to 17 seats, while the unnested ensemble has a range of 7 to 18 seats. Again, the range of possible seat values is nearly identical, particularly given how few of the 7 and 18 seat maps are sampled. Thus, in Ohio, the 3:1 nesting requirement does not significantly impact the number of seats won.

6.2 Comparing Wisconsin Nested and Unnested

We compare the behavior of the nested and unnested ensembles for Wisconsin. See Figure 8. Under the SEN18 data, the nested chain is shifted marginally to the left, in favor of the Republicans, but the range of possible seats is nearly identical in both ensembles. In the nested ensemble, the range is 15 to 24 seats, while in the unnested ensemble it is 15 to 25 seats.

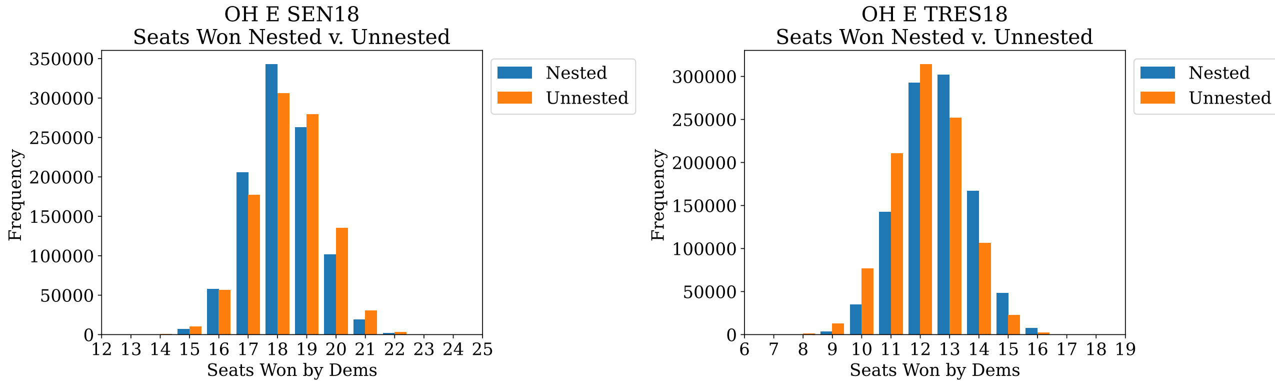

Figure 7: Comparing the nested and unnested ensembles in Ohio under SEN18 and TRES18. Chains were started from the enacted Senate map (E).

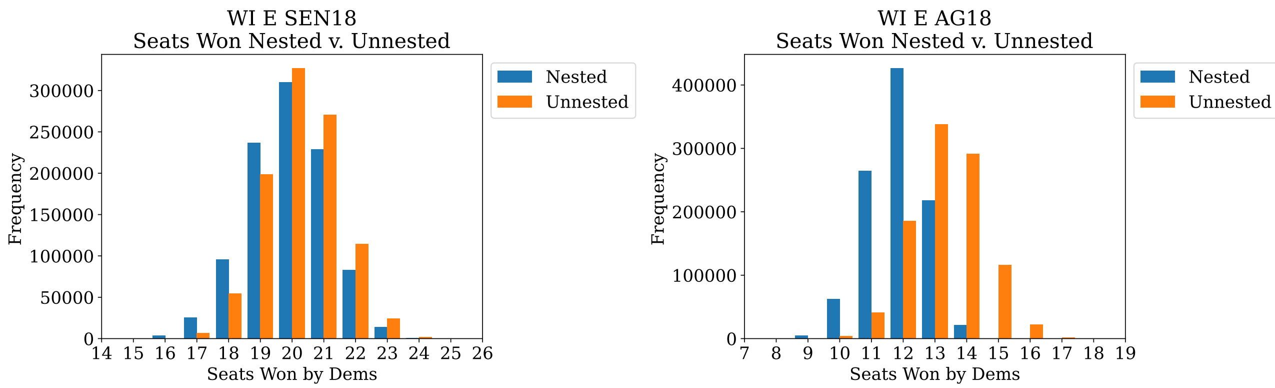

Figure 8: Comparing the nested and unnested ensembles in Wisconsin. Chains were started from the enacted Senate map (E).

The AG18 election data is the only data for which we see a significant change in distribution. Here, the bulk of the nested ensemble is shifted in favor of Republicans, with the range of seats changing from 9 through 18 (unnested) to 8 through 16 (nested). Even with the visible difference in the histograms, we still find that the nesting requirement only reduced the maximum number of seats won by 2. It is interesting to note that in Ohio, the distributions shifted in either direction, but in Wisconsin, the nesting requirement seems to shift the distributions in favor of the Republicans regardless of the election data.

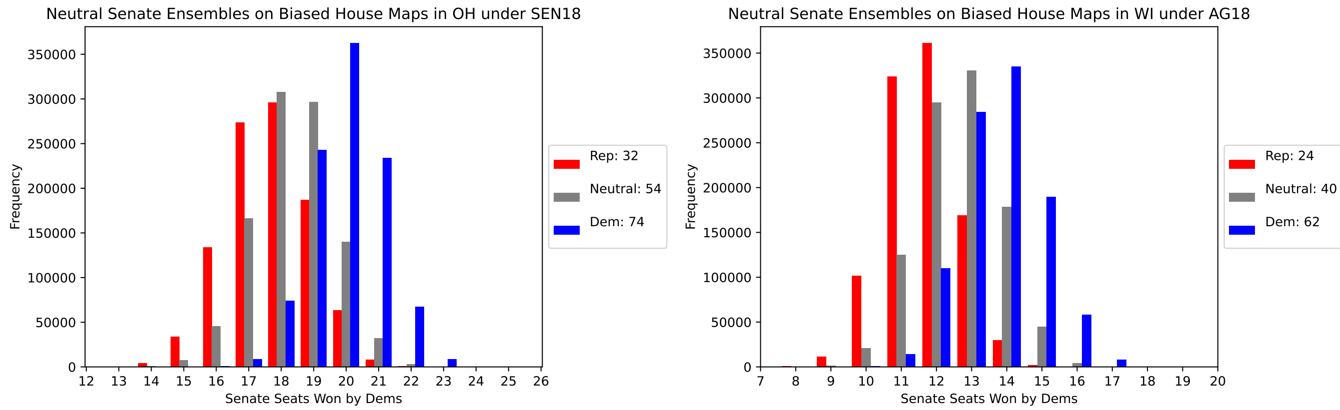

Figure 9: Three neutral Senate distributions generated by Swap ensembles based on three different house maps: one with Republican bias, one with Democrat bias, and one without any bias. The number in the legend indicates the number of seats won by the Democrats at the House level under the chosen House dual graph.

7 Mitigating Partisan Gerrymandering

It is natural to wonder whether the nesting rule mitigates partisan gerrymandering. We first examine what impact the underlying House map has on the distribution of possible Senate maps. In order to study this we used the short burst algorithm to generate House maps with extreme numbers of seats won, both for Democrats and then again for Republicans [5]. We then took these biased House maps as our underlying dual graphs, and generated an ensemble of nested maps to see how this impacted the distribution of Senate seats. We also used a neutral House map as a control.

Despite enormous differences in the number of seats won for Democrats at the House level, the choice of House map had little impact on the distribution of Senate seats. In Figure 9 we see that the distributions built on the Republican House map and the Democrat House map have visibly shifted away from the neutral distribution, in favor of whatever party we were biasing for. Table 1 summarizes the ranges of the ensembles. When we consider just how biased the underlying House maps were and how similar the ranges of the distributions are, this experiment suggests that regardless of how the House map was drawn, there is a wide range of possibilities for the Senate map. This is in some ways expected. If gerrymandering is a process that "happens in the margins," the act of 3:1 nesting tends to erase carefully

	OH SEN18			WI AG18		
House Bias	House D	Min D	Max D	House D	Min D	Max D
Republican	32	13	22	24	8	16
Neutral	54	13	23	40	9	17
Democrat	74	15	25	62	10	19

Table 1: The range of Democrat seats won by the neutral ensemble of Senate maps built on three different House maps. The column “House D” reports the number of seats won by Democrats in each House map.

	Republican		Democrat	
	Unnested Opt.	Nested Opt.	Unnested Opt.	Nested Opt.
OH SEN18	10	12	28	26
OH TRES18	3	5	22	19
WI SEN18	10	13	30	28
WI AG18	5	6	22	21

Table 2: The seats won by Democrats under the most biased map found by nested and unnested optimizations.

drawn boundary lines at the Senate level, resulting in a more neutral distribution.

In addition to generating a neutral distribution on the biased House maps, we also attempt to generate biased Senate maps. We generate a biased Senate map on the precinct dual graph (thus acting as a control), and another biased Senate map on the dual graph of the biased House map we generated above. Again, we use short burst optimization. Table 2 shows that the unnested Senate map optimization consistently found more biased maps than the nested Senate map optimization. This suggests that 3:1 nesting can, to some extent, curtail partisan gerrymandering.

8 Conclusion

In summary, with the exception of the Wisconsin AG18 data, the 3:1 nesting rule seems to have little impact on the distribution of Democrat seats won. This echoes the results of [4].

Even in the case of AG18, the nesting requirement only reduced the maximum number of seats won by 2. We also found that nesting allows a wide range of Senate maps regardless of bias at the House level, as well as reduces the overall ability of a party to optimize for seats won. Taken as a whole, our findings suggest that nesting rules may be used as a policy intervention to mitigate partisan gerrymandering.

Some future directions include:

1. In order to make more realistic Senate maps, it would be useful to implement Ohio's county splitting requirements. One promising avenue for this is to use Forest ReCom, a recently developed Markov chain designed to preserve hierarchical structures, like counties [3].

2. If biasing the House maps for seats did not separate the Senate distributions, what would cause a separation? In other words, what properties of the House map actually impact the possible 3:1 nestings?

3. What happens if you build the Senate map first, and then subdivide each district?

4. How much of a difference is there between using Swap and ReCom to generate nested maps?

8.1 Acknowledgments

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